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A SECOND-ORDER APPROXIMATION TO NATURAL CONVECTION FOR LARGE RAYLEIGH NUMBERS AND SMALL PRANDTL NUMBERS

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SUMMARY

The problem under investigation is that of fluid flow within an enclosed rectangular cavity. It is assumed that one wall is maintained at a constant temperature T_1 (hot wall) and the other wall is maintained at a constant temperature T_0 (cold wall). At the remaining walls, two separate cases are studied. In the first, an adiabatic boundary condition is assumed. That is, the normal derivative of the temperature function is assumed to be 0. In the second, it is assumed the temperature varies linearly from T_0 to T_1.

The purpose of this paper is the application of a second order numerical technique to the problem of fluid flow within a heated closed cavity. The method is a modification of a method developed by Shay[1] and applied to the driven cavity problem. In order to test the viability of this technique, it was decided to extend the technique to the problem of natural convection in a square. Jones[2] proposed that this problem is suitable for testing techniques that may be applied to a wide range of practical problems such as reactor insulation, cooling of radioactive waste containers, solar energy collection and others.[3]

The technique makes use of second-order finite difference approximations to all derivatives in the governing equations. Furthermore, second-order approximations are also used to determine boundary vorticities and, when the adiabatic boundary condition is used, for the boundary temperatures as well. In some works, where second-order approximations are used at interior points, second-order boundary approximations have been sacrificed in favour of a more stable, but first-order boundary approximation.

The current approximations are generated by writing the unknown value of a function at a given interior node as a linear combination of unknown function values at all of the neighbouring nodes. Then the function values at these neighbouring nodes are expanded in a Taylor series about the given node. Through appropriate regrouping of terms and the use of the equations to the solved, constraints are imposed on the coefficients of the linear combination to yield a second-order approximation. As it turns out, there are more unknowns than constraints and, as a result, we are left with some freedom in choosing coefficients. In this work this freedom was used to choose coefficients in such a way as to maximize stability of the resulting system of equations. In other words, the approximations to the governing partial differential equation are individually determined at each point dependent on the direction of flow in order to generate the best possible stability. This idea is analogous to that used in the derivation of the upwind method. However, the current method is second-order accurate where the upwind method is only first-order accurate. Thus, what is generated is an easily implemented second-order method that yields a system of equations that has proved easy to solve.

The system of equations is solved via the method of successive overrelaxation. The stability of the method is shown in the convergence for a wide range of Rayleigh numbers, Prandtl numbers and mesh sizes. Level curves of the stream, vorticity and temperature functions are provided for Rayleigh numbers (Ra) as large as

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100,000, Prandtl numbers (Pr) as small as 0·0001, and mesh sizes as small as 0·0125. Values of the Nusselt number have also been calculated through the use of Simpson's rule, and a second order approximation to the normal derivative of the temperature along the cold wall. Comparisons are made with other current works to aid in the verification of this methods' accuracy and also with the first-order upwind method to demonstrate superiority over the first-order method.

INTRODUCTION

Numerical results for this problem have been previously obtained by others. However, most of the results were obtained for limited values of Ra or Pr, or with first-order numerical approximations. For example, Poots[4] used a numerical method based on the use of orthogonal polynomials. Rosen[5] employed linear programming techniques, and Newell and Schmidt[6] used central difference approximations to first derivative terms. All used a value of $Pr = 0{\cdot}73$. In addition, Poots and Rosen could not obtain convergence for $Ra > 10{,}000$. Others[7–13] have also used central difference approximations. However, central differences tend to be unstable for small values of Pr. For example, Rubel and Landis[10] and Shembharker and Gururaja[11] used values of $Pr \geqslant 1$. de Vahl Davis[7] and Wilkes and Churchill[13] were successful with $Pr \geqslant 0{\cdot}1$. Elder[8] was able to obtain convergence with $Pr = 0{\cdot}01$, but at the expense of severely limited values of Ra (at $Pr = 0{\cdot}01$ stable solutions were obtained only for $Ra < 60$).

Part of the problem with central differences is that the resulting coefficient matrix contains off-diagonal values that are large relative to the diagonal values. Thus, iterating to a solution becomes difficult.[14]

One way to avoid relatively large off-diagonal elements is to use the upwind method developed by Greenspan.[15] Schultz[16] and MacGregor and Emery[17] applied the upwind method to the current problem. The resulting coefficient matrices were diagonally dominant for all values of Ra and Pr. Convergence was reported for values of Ra as large as 100,000 and values of Pr as small as 0·00001. The drawback of this method is its first-order accuracy. Consequently, very small mesh sizes are needed to guarantee accuracy.

More recently, de Vahl Davis and Jones[3] have published a comparison exercise which summarizes contributions from 36 sources along with results from a method they describe in this reference. Some of these contributions produced results for large Rayleigh numbers, some used coarse grids, and some used first-order boundary approximations. In short, although many results for large Rayleigh numbers have been described, there is no evidence that any one of these methods have produced results for large Rayleigh numbers and small Prandtl numbers with small grids *and* second order boundary approximations. The current method has produced such results.

THE PROBLEM

The problem is described in Reference 16. The equations to be satisfied in the interior of the region (Figure 1) are as follows:

$$\nabla^2\psi = -\zeta \tag{1}$$

$$\nabla^2 T + \psi_x T_y - \psi_y T_x = 0 \tag{2}$$

and

$$\nabla^2\zeta + (1/Pr)(\psi_x\zeta_y - \psi_y\zeta_x) + Ra\,T_y = 0 \tag{3}$$

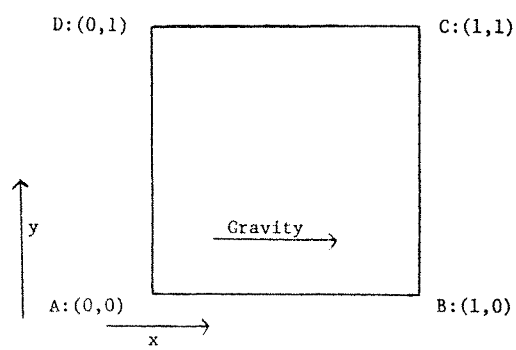

Figure 1.

The stream, vorticity and normalized temperature functions are represented by ψ, ζ and T, respectively. The Rayleigh and Prandtl numbers are given by Ra and Pr, respectively.

Boundary conditions for the problem are

$$\psi = 0 \quad \text{on} \quad \text{ABCDA} \tag{4}$$

$$\partial\psi/\partial y = 0, \quad T = 0 \text{ on AB} \tag{5a}$$

$$\partial\psi/\partial x = 0, \quad \partial T/\partial x = 0 \text{ on AD and BC} \tag{5b}$$

$$\partial\psi/\partial y = 0, \quad T = 1 \text{ on CD} \tag{5c}$$

These are boundary conditions for the case where the surfaces between the hot and cold walls are insulated. In the case where the temperature varies linearly along the walls separating the hot and cold surfaces, the condition $\partial T/\partial x = 0$ is replaced by $T = y$ in equation (5b). This case is considered later.

DIFFERENCE EQUATIONS

To start the method, a rectangular array of nodes is placed over the region in Figure 1. It is assumed the vertical and horizontal spacings are equal and are described by h. Define the inner boundary as the collection of all points that lie a distance of h from the boundary. Values at these grid points are used to guarantee that the normal derivative conditions of the stream function are satisfied.

Boundary vorticities may be approximated by (Figure 2).[18]

$$\zeta_0 = -3\psi_1/h^2 - \zeta_1/2 + O(h^2). \tag{6}$$

A similar derivation shows that boundary temperature may be approximated by

$$T_0 = 4T_1/3 - T_2/3 \tag{7}$$

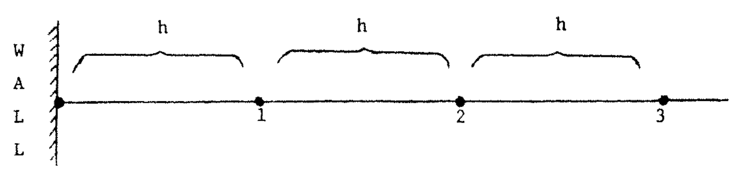

Figure 2.

Stream values on the inner boundary are determined by first writing.[19]

$$\psi_{n_0} = (-11\psi_0 + 18\psi_1 - 9\psi_2 + 2\psi_3)/6h + O(h^3) \tag{8}$$

Here, ψ_{n_0} represents the derivative of ψ in the direction normal to the wall at the wall. Since $\psi_{n_0} = 0$ and $\psi_0 = 0$, ψ_1 is expressed as

$$\psi_1 = \psi_2/2 - \psi_3/9 + O(h^4) \tag{9}$$

The higher order approximation (8) is used so that when (9) is used in (6) the $O(h^2)$ accuracy of (6) is maintained.

At any interior node (Figure 3) $\nabla^2\psi$ may be approximated by

$$\nabla^2\psi = (-4\psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4)/h^2 + O(h^2) \tag{10}$$

Similarly we have

$$\nabla^2\zeta = (-4\zeta_0 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4)/h^2 + O(h^2) \tag{11}$$

and

$$\nabla^2 T = (-4T_0 + T_1 + T_2 + T_3 + T_4)/h^2 + O(h^2) \tag{12}$$

In (3), T_y is approximated by

$$T_y = (T_2 - T_4)/2h + O(h^2) \tag{13}$$

To approximate $(1/Pr)\psi_x\zeta_y - (1/Pr)\psi_y\zeta_x$ in (3), write

$$(1/Pr)\psi_x\zeta_y - (1/Pr)\psi_y\zeta_x|_{P_0} = \sum_{i=0}^{8} \alpha_i\zeta_i \tag{14}$$

where the α_i are to be determined. Expand each ζ_i in a Taylor series about the point P_0.

Next, reorganize terms of the expansion and group together the coefficients of ζ_0, ζ_{x_0}, ζ_{y_0}, etc., up to third partial derivatives. Then equate with coefficients of like terms in (14). Therefore in order for $\sum_{i=0}^{8}\alpha_i\zeta_i$ to approximate $(1/Pr)\psi_x\zeta_y - (1/Pr)\psi_y\zeta_x$ it is sufficient that the α_i satisfy

$$\sum_{i=0}^{8} \alpha_i = 0 \tag{15}$$

$$\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = -\psi_y/(hPr)|_{P_0} \tag{16}$$

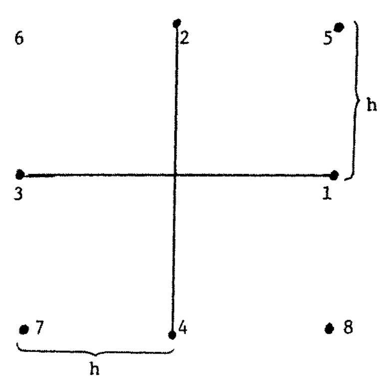

Figure 3.

$$\alpha_2 - \alpha_4 + \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = \psi_x/(hPr)|_{P_0} \tag{17}$$
$$\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0 \tag{18}$$
$$\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8 = 0 \tag{19}$$
$$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0 \tag{20}$$
$$\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = 0 \tag{21}$$
$$\alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0 \tag{22}$$
$$\alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = 0 \tag{23}$$
$$\alpha_2 - \alpha_4 + \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0 \tag{24}$$

However, it is generally not possible to satisfy (21) and (24) if (16) and (17) are satisfied. As a result, (21) and (24) will constitute an error term of the form

$$(-\psi_y/Prh)\zeta_{xxx}(h^3/6)|_{P_0} + (\psi_x/Prh)\zeta_{yyy}(h^3/6)|_{P_0} = (h^2/6Pr)(\psi_x\zeta_{yyy} - \psi_y\zeta_{xxx})|_{P_0}$$

In addition, (22) and (23) are dropped since this yields more flexibility in determining the α_i and does not introduce any lower order error terms. In fact, the six remaining equations in nine unknowns result in three degrees of freedom with which to chose the α_i. If α_6, α_7 and α_8 are the independent choices then

$$\alpha_0 = 2(\alpha_6 + {}_8) \tag{25}$$
$$\alpha_1 = ((-1/Pr)\psi_y/2h)|_{P_0} - \alpha_6 + \alpha_7 - 2\alpha_8 \tag{26}$$
$$\alpha_2 = ((1/Pr)\psi_x/2h)|_{P_0} - 2\alpha_6 + \alpha_7 - \alpha_8 \tag{27}$$
$$\alpha_3 = ((1/Pr)\psi_y/2h)|_{P_0} - \alpha_6 - \alpha_7 \tag{28}$$
$$\alpha_4 = ((-1/Pr)\psi_x/2h)|_{P_0} - \alpha_7 - \alpha_8 \tag{29}$$
$$\alpha_5 = \alpha_6 - \alpha_7 + \alpha_8 \tag{30}$$

So, for any choice of α_6, α_7 and α_8, equation (14) becomes a second order approximation with an error of

$$((h^2(1/Pr)/6)(\psi_x\zeta_{yyy} - \psi_y\zeta_{xxx}) + h^3(\zeta_{xxy}(\alpha_6 - \alpha_7) + \zeta_{xyy}(\alpha_8 - \alpha_7)))|_{P_0} \tag{31}$$

At each interior node, P_0, make the following assignments.

If $\psi_x|_{P_0} \geqslant 0$ and $\psi_y|_{P_0} \geqslant 0$ let

$$\alpha_6 = 0; \alpha_7 = ((-1/Pr)\psi_x/2h)|_{P_0}; \alpha_8 = ((-1/Pr)(\psi_x + \psi_y)/2h)|_{P_0} \tag{32}$$

If $\psi_x|_{P_0} < 0$ and $\psi_y|_{P_0} < 0$ let

$$\alpha_6 = ((1/Pr)(\psi_x + \psi_y)/2h)|_{P_0}; \alpha_7 = ((1/Pr)\psi_y/2h)|_{P_0}; \alpha_8 = 0 \tag{33}$$

If $\psi_x|_{P_0} \geqslant 0$ and $\psi_y|_{P_0} < 0$ let

$$\alpha_6 = ((1/Pr)\psi_y/2h)|_{P_0}; \alpha_7 = ((1/Pr)(\psi_y - \psi_x)/2h)|_{P_0}; \alpha_8 = ((-1/Pr)\psi_x/2h)|_{P_0} \tag{34}$$

If $\psi_x|_{P_0} < 0$ and $\psi_y|_{P_0} \geqslant 0$ let

$$\alpha_6 = ((1/Pr)\psi_x/2h)|_{P_0}; \alpha_7 = 0; \alpha_8 = ((-1/Pr)\psi_y/2h)|_{P_0} \tag{35}$$

Corresponding values of α_5 are

$$((-1/Pr)\psi_y/2h)|_{P_0}; ((1/Pr)\psi_x/2h)|_{P_0}; 0; ((1/Pr)(\psi_x - \psi_y)/2h)|_{P_0} \tag{36}$$

respectively.

In all cases

$$\alpha_0 = ((-1/Pr)(|\psi_x| + |\psi_y|)/h)|_{P_0} \tag{37}$$

and

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\alpha_0/2 \tag{38}$$

The lowest order error term is

$$E = (h^2/6Pr)(\psi_x\zeta_{yyy} + 3\psi_x\zeta_{xxy} - 3\psi_y\zeta_{xyy} - \psi_y\zeta_{xxx})|_{P_0} \tag{39}$$

If $\psi_x|_{P_0}$ and $\psi_y|_{P_0}$ are approximated by $(\psi_1 - \psi_3)/2h$ and $(\psi_2 - \psi_4)/2h$, respectively, in each of the α_i, the complete approximation of (3) is

$$(-4 + \Omega_0)\zeta_0 + \sum_{i=1}^{4} (1 + \Omega_i)\zeta_i + \sum_{i=5}^{8} \Omega_i\zeta_i + Ra(T_2 - T_4)h/2 = 0 \tag{40}$$

where

$$\Omega_0 = -(|\psi_1 - \psi_3| + |\psi_2 - \psi_4|)/(2Pr) \tag{41}$$

$$\Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = -\Omega_0/2 \tag{42}$$

and

$$\Omega_i = h^2\alpha_i \text{ for } i = 5, 6, 7, 8 \tag{43}$$

This method of approximation is similar to the upwind method in that coefficients are chosen based on the signs of ψ_x and ψ_y. As a result, contributions to diagonal elements of the coefficient matrix from the term $(1/Pr)(\psi_x\zeta_y - \psi_y\zeta_x)$ are guaranteed to be negative, thus adding to the magnitude of the -4 generated in the approximation of $\nabla^2\zeta$. In fact, it is not difficult to show that, in each case $\sum_{i=1}^{8}|\alpha_i|/|\alpha_0| = 3$ for any value of Pr.

To complete the description of the method, $\psi_x T_y - \psi_y T_x$ from (2) must be approximated. To do this, simply replace $(1/Pr)$ with 1 and ζ with T and repeat the previous development.

Given the complete discretization of (1)–(3), the method is started by assigning initial values of ψ, ζ and T to each node. Initial values may consist of all 0s, results of another method, or results from the current method run with different values of h, $(1/Pr)$ or Ra. Then successive sweeps of the region are made, redefining the stream, vorticity, and temperature functions at each node through the use of the successive overrelaxation (S.O.R.) technique until all three functions have converged. The order in which nodes are covered within a sweep may affect stability. In an application of a similar method to another problem[19] divergence resulted when, with each sweep, the nodes were covered from left to right along each row, starting at the bottom row and proceeding to the top row. However, if in alternate sweeps the pattern was reversed, convergence resulted.

RESULTS AND CONCLUSIONS

Coverged solutions have been obtained for Rayleigh numbers up to 100,000 and Prandtl numbers as small as 0·0001. Results have been obtained for a decreasing sequence of mesh sizes, and from the use of extrapolation on results of the two smallest mesh sizes ($h = 0{\cdot}025$ and $h = 0{\cdot}0125$). The extrapolated results are compared with results considered to be the best known generated by de Vahl Davis[3] (Table I). As in Reference 3, the average Nusselt number was calculated through the use of a three point approximation to $\partial T/\partial y$ at the cold wall and Simpson's rule to approximate $\int_0^1(\partial T/\partial y)|_{y=0}\,dx$. Fourth order approximations were used to calculate the maximum vertical velocity on the horizontal midplane (v_{max}), maximum horizontal velocity on the vertical midplane (u_{max}), and the maximum and minimum local Nusselt numbers (Nu_{max} and Nu_{min}). Within Table I, x or y indicates a co-ordinate of a point where the value immediately above was located. The maximum stream value is given by ψ_{max} and ψ_{mid} represents the stream value at the midpoint. There is excellent agreement between the results, with relative differences generally less than 1 per cent (see Table I).

Figures 4–6 contain level curves for the stream, vorticity and temperature functions.

Table I. Comparison of results from the current method and Reference 3. $Pr = 0{\cdot}71$.

	$Ra = 10{,}000$		$Ra = 100{,}000$	
	Reference 3	Current study	Reference 3	Current study
Nu	2·238	2·257	4·505	4·505
Nu_{max}	3·527	3·562	7·717	7·793
x	0·143	0·125	0·082	0·075
Nu_{min}	0·586	0·574	0·729	0·723
x	1·0	1·0	1·0	1·0
v_{max}	−16·178	−16·153	−34·77	−34·71
x	0·823	0·825	0·854	0·85
u_{max}	19·643	19·608	68·25	68·49
y	0·119	0·125	0·066	0·059
ψ_{mid}	5·079	5·070	9·120	9·089
ψ_{max}	n.a.	5·070	9·622	9·591
x	n.a.	0·5	0·399	0·4
y	n.a.	0·5	0·713	0·709

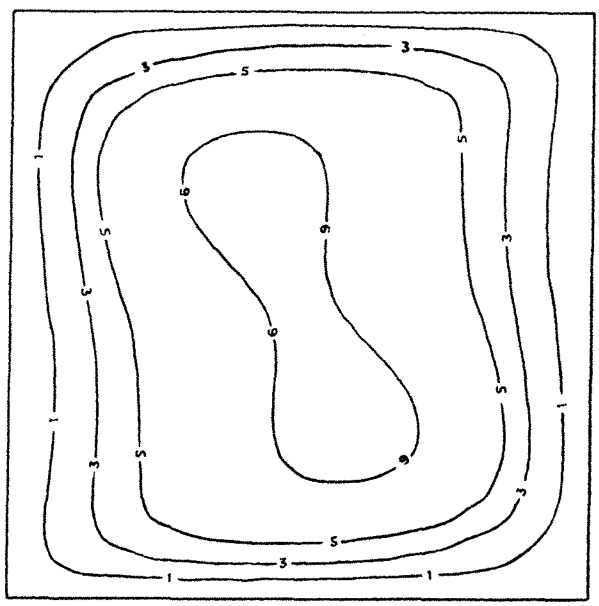

Figure 4. Streamlines for Rayleigh number 100,000, Prandtl number 0·71 and mesh size 0·0125

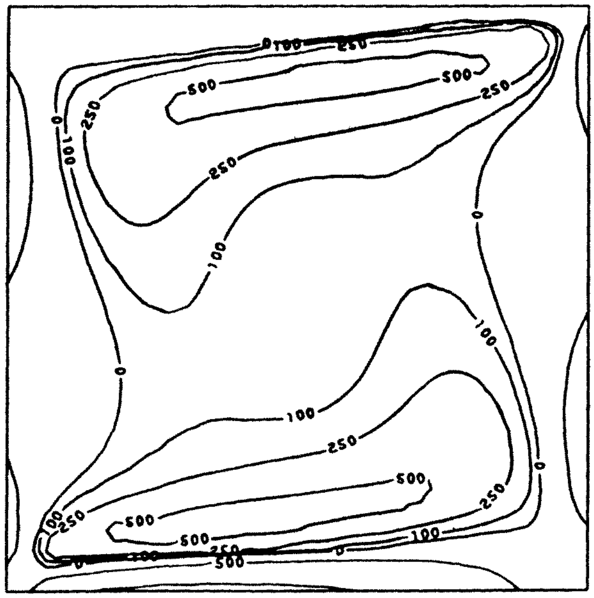

Figure 5. Vorticities for Rayleigh number 100,000, Prandtl number 0·71 and mesh size 0·0125

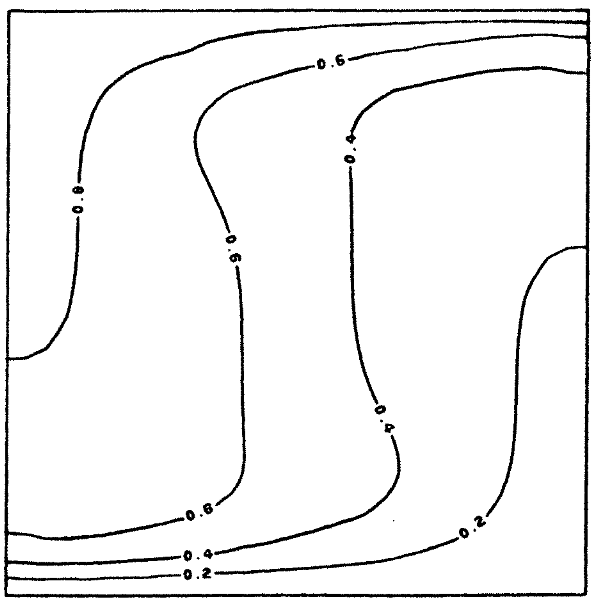

Figure 6. Temperature for Rayleigh number 100,000, Prandtl number 0·71 and mesh size 0·0125

It is noteworthy to mention that despite the second-order accuracy of this method, it is shown that a very fine mesh size is quite necessary in order to achieve excellent accuracy when large Rayleigh numbers are used. This is demonstrated in Table II where results are displayed as functions of h.

As previously stated, the method has also proved stable for small values of the Prandtl number.

Table II. Results for $Ra = 100{,}000$, $Pr = 0{\cdot}71$ as functions of h. Extrapolated results are obtained from results with $h = 0{\cdot}025$ and $h = 0{\cdot}0125$

	$h = 0{\cdot}05$	$h = 0{\cdot}025$	$h = 0{\cdot}0125$	extrapolated
$\overline{Nu}$	4·943	4·658	4·543	4·505
Nu_{max}	7·744	7·877	7·814	7·793
x	0·1	0·075	0·075	0·075
Nu_{min}	0·597	0·672	0·710	0·723
x	1·0	1·0	1·0	1·0
v_{max}	− 33·67	− 34·40	− 34·63	− 34·71
x	0·85	0·85	0·85	0·85
u_{max}	67·27	69·28	68·69	68·49
y	0·1	0·075	0·063	0·059
ψ_{mid}	9·31	9·24	9·127	9·089
ψ_{max}	9·83	9·74	9·628	9·591
x	0·4	0·4	0·4	0·4
y	0·7	0·725	0·713	0·709

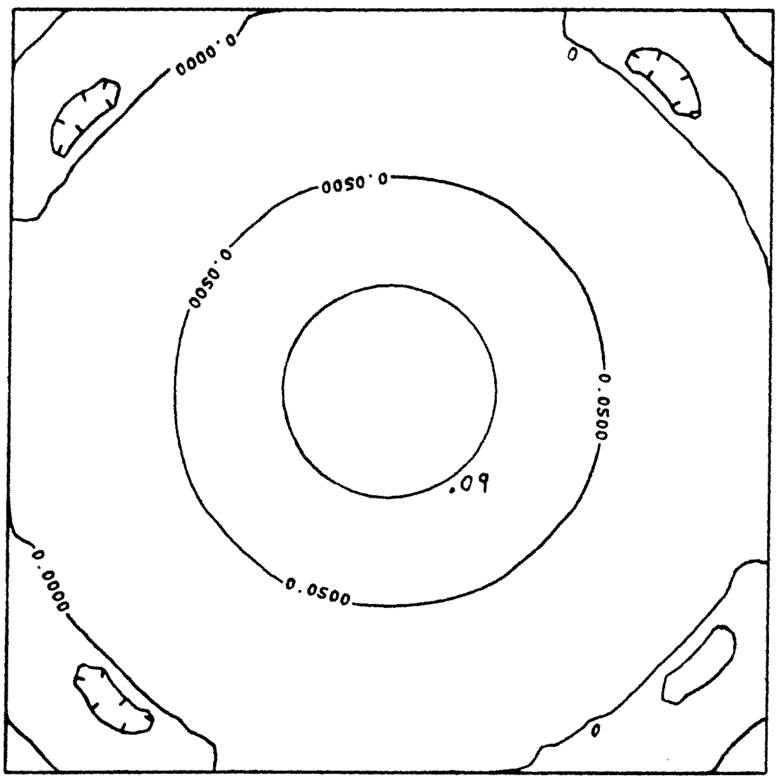

Figure 7. Streamlines for Rayleigh number 100, Prandtl number 0·0001 and mesh size 0·0125

Table III. Results for $Ra = 100$, $Pr = 0{\cdot}0001$ and $h = 0{\cdot}0125$

Nu	1·00045
(Nu_{max}, x)	(1·0312,0)
(Nu_{min}, x)	(0·9688,1)
(v_{max}, x)	(− 0·3196, 0·787)
(u_{max}, y)	(0·3196, 0·213)
ψ_{mid}	0·1054
ψ_{max}	0·1054

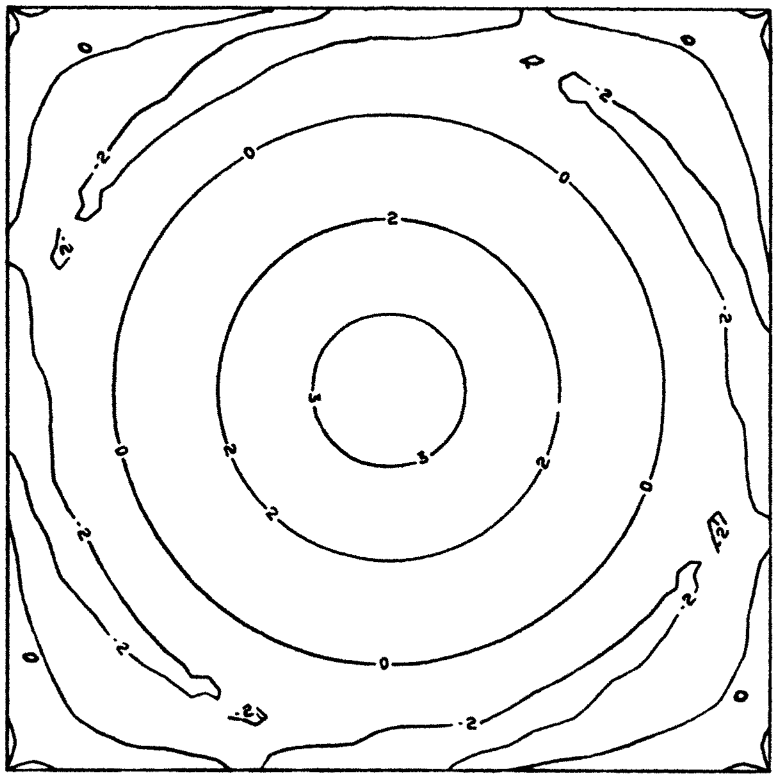

Figure 8. Vorticities for Rayleigh number 100, Prandtl number 0·0001 and mesh size 0·0125

Table IV. Comparison of results from the current method and the first-order method in Reference 16. $Ra = 10{,}000$ and $Pr = 0{\cdot}73$

h	First-order method ψ_{mid}	First-order method ζ_{mid}	Current method ψ_{mid}	Current method ζ_{mid}
0·1	7·962	168·0	6·764	139·4
0·05	7·066	142·0	6·430	130·7
0·025	6·590	135·0	6·357	129·4
0·0125	6·423	132·3	—	—

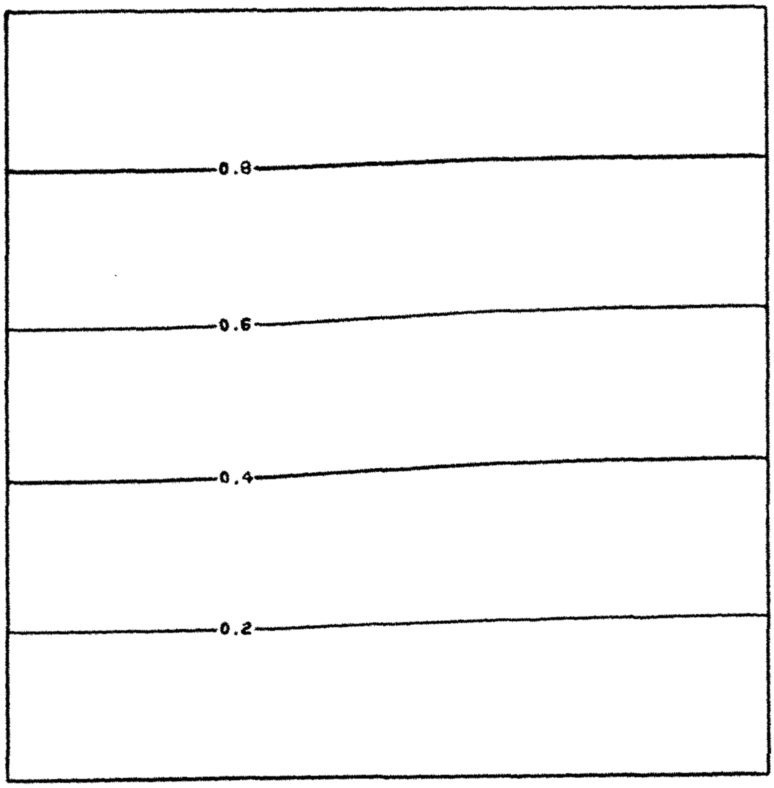

Figure 9. Temperatures for Rayleigh number 100, Prandtl number 0·0001 and mesh size 0·0125

Table III contains a summary of results obtained for $Pr = 0{\cdot}0001$ and $Ra = 100$. Figures 7–9 contain level curves for the stream, vorticity and temperature. A mesh size of 0·0125 was used.

In addition to the displayed results from Reference 3, a collection of results from 36 outside sources is also summarized in Reference 3. However, none of these results were obtained with as fine a grid for $Ra = 100{,}000$ as were the current results. One source did use a similar size mesh for $Ra = 1{,}000{,}000$ but had difficulties preserving the symmetry of the problem. Also, none of the methods indicate success with small Prandtl numbers. It is the success of the current method with the wide range of both Rayleigh and Prandtl numbers (due primarily to the 3 degrees of freedom in choosing coefficients of approximating equations) at small mesh sizes that indicate the potential of this method as an accurate technique applicable to a wide variety of problems. Other features of the current method include the second-order accuracy of all approximating equations (including boundary approximations—some authors used first-order boundary approximations), the use of grid refinement and extrapolation on an already fine mesh size, easy implementation on a computer, and flexibility for adaption to other types of problems. None of the methods in Reference 3 indicate possession of all of the above features.

The method was also tested for the case of linear temperature variation between the hot and cold walls. This was done primarily to compare with existing results generated by the first-order upwind method. Table IV contains a comparison between results of the current method and results from Reference 16.

The superiority of the second-order method is shown here. For example, results comparable to those obtained by the first order method with a mesh size of 0·0125 are comparable to results obtained by the second order method with a mesh size of only 0·05. In fact, these second-order results are probably better.

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